Fourier transform of function on locally compact Abelian groups taking value in Banach spaces *

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Abstract

We consider Fourier transform of vector-valued functions on a locally compact group G, which take value in a Banach space X, and are square-integrable in Bochner sense. If G is a finite group then Fourier transform is a bounded operator. If G is an infinite group then Fourier transform $F : L_2(G, X) \to L_2(\widehat{G}, X)$ is a bounded operator if and only if Banach space X is isomorphic to a Hilbert one.

1 Fourier transform over groups R, Z, T

In the paper [1] J. Peetre proved an extension of Hausdorf–Young's theorem describing image of $L_q(\mathbf{R})$ under Fourier transform. He considered vector-valued $x \in L_q(\mathbf{R}, X)$, $1 \le q \le 2$ on the real axis taking value in Banach space X, and integrable in Bochner sense, i.e. weakly measurable with finite norm

$$||x||_{L_q(\mathbf{R},X)} = \Big(\int_{\mathbf{R}} ||x(t)||_X^q dt\Big)^{1/q}.$$

J. Peetre noted, that for $q = 2$ in all known to him cases Fourier transform

$$\mathcal{F} : L_2(\mathbf{R}, X) \to L_2(\mathbf{R}, X), \quad (\mathcal{F}x)(s) = \int_{\mathbf{R}} x(t)e^{-2\pi ist}dt.$$

was bounded only if X was isomorphic to a Hilbert space. In [3] Polish mathematician S. Kwapien in fact proved the following

Theorem 1 *Statements below are equivalent:*
1) *Banach space X is isomorphic to a Hilbert one.*
2) *There exists $C > 0$ such that for any positive integer n and $x_0, x_1, x_{-1}, \ldots, x_n, x_{-n} \in X$*

$$\int_0^1 \Big\| \sum_{k=-n}^{n} e^{2\pi ikt} \cdot x_k \Big\|^2 dt \le C \sum_{k=-n}^{n} \|x_k\|^2.$$

3) *There exists $C > 0$ such that for any positive integer n and $x_0, x_1, x_{-1}, \ldots, x_n, x_{-n} \in X$*

$$C^{-1} \sum_{k=-n}^{n} \|x_k\|^2 \le \int_0^1 \Big\| \sum_{k=-n}^{n} e^{2\pi ikt} \cdot x_k \Big\|^2 dt.$$

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4) *Fourier transform $\mathcal{F}$ initially defined on a dense subspace of simple functions $D_{\mathcal{F}} \subset L_2(\mathbf{R}, X)$,*

$$D_{\mathcal{F}} = \Big\{ x(t) = \sum_{k=1}^{n} I_{A_k}(t) \cdot x_k \Big\},$$

is a bounded operator. Here A_k are disjoint subset of finite measure in $\mathbf{R}$, I_{A_k} are indicators (i.e. functions equal to 1 on A_k and to 0 elsewhere), x_k are vectors in X.

Let us define Fourier transform of a vector-valued function over integers $\mathbf{Z}$ by

$$\mathcal{F}_{\mathbf{Z}} : L_2(\mathbf{Z}, X) \to L_2(\mathbf{T}, X) : (x_k) \mapsto \sum_{k \in \mathbf{Z}} e^{-2\pi i k t} \cdot x_k,$$

Here $\mathbf{T}$ denotes a one-dimensional torus $\mathbf{T} = \mathbf{R}/\mathbf{Z}$, which is isomorphic to $[0, 1]$ with the length in the cathegory of spaces with a measure.

Statement 2) in Theorem 1 means that $\mathcal{F}_{\mathbf{Z}} \mathcal{I}_{\mathbf{Z}}$ is a bounded operator (here $\mathcal{I}_{\mathbf{Z}}$ denotes the operator of changing variable $\mathcal{I}_{\mathbf{Z}} : (x_k) \mapsto (x_{-k})$, which is an isometry). That is why $\mathcal{F}_Z$ is bounded on a dense subspace of $L_2(\mathbf{Z}, X)$ consisting of compactly-supported functions, and can be continued to the whole $L_2(\mathbf{Z}, X)$.

Statement 3) in Theorem 1 means that inverse Fourier transform

$$\mathcal{F}_{\mathbf{Z}}^{-1} : L_2(\mathbf{T}, X) \to L_2(\mathbf{Z}, X)$$

is a bounded operator.

2 Generalization and necessary facts

Fourier transform of Banach space-valued functions on a group different from $\mathbf{R}$,$\mathbf{Z}$,$\mathbf{T}$ (namely on additive group of p-adic field $\mathbf{Q}_p$) was considered in [4].

Now it's natural to look at the general case of arbitrary locally compact group G. We consider functions on G taking value in Banach space X, that are square-integrable in Bochner sense, and Fourier transform

$$\mathcal{F} \equiv \mathcal{F}_G : L_2(G, X) \to L_2(\widehat{G}, X), \quad (\mathcal{F}x)(\xi) = \int_G \langle \xi, t \rangle_G x(t) d\mu_G(t).$$

Here $\widehat{G}$ is Pontryagin dual to G (group of characters), $\langle \xi, t \rangle_G$ is the canonical pairing between $\widehat{G}$ and G, μ_G is Haar measure.

First we recall some necessary results. We refer to [5] for results in harmonic analysis, and to [6] for structure theory of locally compact groups, Bruhat–Schwartz theory is exposed in [7].

Fix a dual Haar measure $\mu_{\widehat{G}}$ on $\widehat{G}$ such that scalar Steklov–Parseval's equality holds

$$||\varphi||^2_{L_2(G)} = \int\limits_G |\varphi|^2 d\mu_G = \int\limits_{\widehat{G}} |\mathcal{F}\varphi|^2 d\mu_{\widehat{G}} = ||\mathcal{F}\varphi||_{L_2(\widehat{G})}.$$

Let $\mathcal{S}(G)$ denote Bruhat–Schwartz space of "smooth fastly decreasing" functions on G.

It's useful to take a dense subspace

$$D_{\mathcal{F}} = L_2(G) \otimes X \subset L_2(G, X),$$

as initial domain of $\mathcal{F}$, where it acts by

$$\mathcal{F}\Big(\sum_{k=1}^{n} \varphi_k(t) \cdot x_k\Big) = \sum_{k=1}^{n} ((\mathcal{F}\varphi_k)(\xi) \cdot x_k).$$

To show denseness of $D_{\mathcal{F}}$ and denseness of $\mathcal{S}(G) \otimes X \subset L_2(G) \otimes X$ consider indicator I_A of arbitrary measurable subset of finite measure (i.e. functions equal to 1 on A and to 0 elsewhere). Clearly, $I_A \in L_2(G)$, and it can be approximated by elements of $\mathcal{S}(G)$ using convolution with any δ-net consisting of Bruhat–Schwartz functions. By definition of $L_2(G, X)$ finite linear combinations

$$\sum_{k=1}^{n} I_{A_k}(t) \cdot x_k \in L_2(G) \otimes X \equiv D_{\mathcal{F}}$$

are dense in $L_2(G, X)$. Here A_k are measurable disjoint subsets of finite measure in G, I_{A_k} are indicators, and $x_k \in X$.

Due to the fact that scalar Fourier transform is a bijection from $L_2(G)$ into $L_2(\widehat{G})$, and is a bijection from $\mathcal{S}(G)$ into $\mathcal{S}(\widehat{G})$, the restriction of vector-valued Fourier transform onto $L_2(G) \otimes X$ is a bijection into $L_2(\widehat{G}) \otimes X$, and its restriction onto $\mathcal{S}(G) \otimes X$ is a bijection to $\mathcal{S}(\widehat{G}) \otimes X$.

To handle the case of infinite group G we need a theorem describing structure of locally compact Abelian groups [6].

Theorem 2 *Let G be a locally compact Abelian group. Then G is a union of open compactly generated subgroups H. Topology of G is the topology of inductive limit. In turn, each compactly-generated subgroup $H \subset G$ is a projective limit of elementary factor-groups H/K, where $K \subset H$ are compact. "Elementary" here means that H/K is isomorphic to cartesian product*

$$H/K \cong \mathbf{R}^{a_{H,K}} \times \mathbf{T}^{b_{H,K}} \times \mathbf{Z}^{c_{H,K}} \times \mathbf{F}_{H,K},$$

where $a_{H,K} \geq 0$, $b_{H,K} \geq 0$, $c_{H,K} \geq 0$, and $\mathbf{F}_{H,K}$ is a finite group.

Definition 1 *We say that G contains an $\mathbf{R}$-component if for some H, K in Theorem 2 number $a_{H,K}$ is positive in elementary factor-group $\mathbf{R}^{a_{H,K}} \times \mathbf{T}^{b_{H,K}} \times \mathbf{Z}^{c_{H,K}} \times \mathbf{F}_{H,K}$.*

In the same way we use phrases "group G contains $\mathbf{Z}$-component", "group G contains $\mathbf{T}$-component".

Now recall some properties of Pontryagin duality.

Consider an exact sequence of homomorphisms (i.e. image of each homomorphism coincides the kernel of the following one)

$$0 \to K \to G \to G/K \to 0, \quad 0 \to H \to G \to G/H \to 0,$$

where K is compact, and H is open. Dual groups $\widehat{G/K}$, $\widehat{G/H}$ can be identified with annihilators

$$K_G^{\perp} = \{\chi \in \widehat{G} : \ \forall g \in K, \ \langle \chi, g \rangle = 1\}, \quad H_G^{\perp} = \{\chi \in \widehat{G} : \ \forall g \in H, \ \langle \chi, g \rangle = 1\}.$$

Moreover, $K_G^{\perp}$ is an open subgroup, and $H_G^{\perp}$ is compact. One has dual oppositely-directed exact sequences

$$0 \leftarrow \widehat{G}/K_G^{\perp} \leftarrow \widehat{G} \leftarrow K_G^{\perp} \leftarrow 0, \quad 0 \leftarrow \widehat{G}/H_G^{\perp} \leftarrow \widehat{G} \leftarrow H_G^{\perp} \leftarrow 0.$$

If $K \subset H$, then $K_G^{\perp} \supset H_G^{\perp}$.

It's not hard to see that Fourier transform of a function on G, that is supported in open subgroup H and is constant on cosets of compact subgroup $K \subset H$, is a function on $\widehat{G}$ supported in $K_G^{\perp}$ and constant on cosets of $H_G^{\perp}$.

3 The case of an arbitrary local compact group

We will prove some necessary lemmas before formulating the main result.

Lemma 1 *Assume that Banach space X is isomorphic to a Hilbert one, i.e. there exists inner product $(\cdot,\cdot)_X$ on X such that for some $C>0$ the following inequality is true*

$$C^{-1}(x,x)_X^{1/2} \le ||x||_X \le C(x,x)_X^{1/2}.$$

Then Fourier transform $\mathcal{F}: L_2(G,X) \to L_2(\widehat{G},X)$ is bounded.

Proof. Consider vector-valued Steklov–Parseval's equality

$$(\mathcal{F}\varphi,\mathcal{F}\varphi)_{L_2(\widehat{G},X)} = \int_{\widehat{G}} (\mathcal{F}\varphi(\xi),\mathcal{F}\varphi(\xi))d\mu_{\widehat{G}}(\xi) = \int_G (\varphi(t),\varphi(t))d\mu_G(t) = (\varphi,\varphi)_{L_2(G,X)},$$

which can be easily checked for $\varphi \in L_2(G)\otimes X$ by means of axioms for inner product, scalar Steklov–Parseval's equality and cross-norm's property

$$(\varphi_1\otimes x_1, \varphi_2\otimes x_2)_{L_2(G,X)} = (\varphi_1,\varphi_2)_{L_2(G)}\cdot(x_1,x_2)_X.$$

Then we have on a dense subspace

$$||\mathcal{F}\varphi||_{L_2(\widehat{G},X)} \le C(\mathcal{F}\varphi,\mathcal{F}\varphi)_{L_2(\widehat{G},X)} = C(\varphi,\varphi)_{L_2(G,X)} \le C^2||\varphi||_{L_2(G,X)},$$

so we can extend $\mathcal{F}$ by continuity onto the whole $L_2(G,X)$. ◀

If G is a finite group, then space $L_2(G,X)$ is isomorphic to the finite Cartesian product X^G. Pontryagin's dual group $\widehat{G}$ is isomorphic to G itself. Self-dual Haar measure possesses the property $\mu_G(G)=\sqrt{|G|}$. Fourier transform, also known as discrete Fourier transform, becomes

$$(\mathcal{F}\varphi)(\xi) = \frac{1}{\sqrt{|G|}}\sum_{t\in G}\langle\xi,t\rangle\varphi(t).$$

Theorem 3 *If G is a finite group, then Fourier transform $\mathcal{F}: L_2(G,X)\to L_2(\widehat{G},X)$ is bounded for any Banach space X.*

Proof. It follows from inequality

$$||\mathcal{F}\varphi||^2_{L_2(\widehat{G},X)} = \sum_{\xi\in\widehat{G}}\Big\|\frac{1}{\sqrt{|G|}}\sum_{t\in G}\langle\xi,t\rangle\varphi(t)\Big\|_X^2 \le \frac{1}{|G|}\sum_{\xi\in\widehat{G}}\Big(\sum_{t\in G}|\langle\xi,t\rangle|\cdot\|\varphi(t)\|_X\Big)^2 =$$
$$= \frac{1}{|G|}\sum_{\xi\in\widehat{G}}\Big(\sum_{t\in G}\|\varphi(t)\|_X\Big)^2 = \Big(\sum_{t\in G}\|\varphi(t)\|_X\Big)^2 \le |G|\sum_{t\in G}\|\varphi(t)\|_X^2 = |G|\cdot\|\varphi\|^2_{L_2(G,X)}.$$

◀

Now pass to infinite groups.

Lemma 2 *Let group G contain* **R***-component,* **T***-component or* **Z***-component. Then boundedness of Fourier transform*

$$\mathcal{F}: L_2(G,X)\to L_2(\widehat{G},X)$$

implies isomorphism of Banach space X to a Hilbert one.

Proof. Consider the case, when group G contains $\mathbf{R}$-component. There are open compactly generated subgroup H in G and compact subgroup $K \subset H$ such that $H/K \cong \mathbf{R}^a \times \mathbf{T}^b \times \mathbf{Z}^c \times F$, where $a \geq 1$.

Let $\tau_1 : H \to H/K$ be a canonical projection, $\tau_2 : H/K \to \mathbf{R}$ be the projection on the first coordinate of $\mathbf{R}^a$, $\tau = \tau_2 \circ \tau_1$.

Consider helper functions $\psi_{\mathbf{R},i} \in L_2(\mathbf{R})$, $2 \leq i \leq a$, $\psi_{\mathbf{T},j} \in L_2(T)$, $1 \leq j \leq b$, $\psi_{\mathbf{Z},k} \in L_2(\mathbf{Z})$, $1 \leq k \leq c$, $\psi_F \in L_2(F)$, each of them having norm equal to 1 in corresponding space. Consider injection $J : L_2(\mathbf{R}, X) \to L_2(H, X)$,

$$J : \varphi \mapsto \left((\varphi \circ \tau) \otimes \left(\bigotimes_{i=2}^{a} \psi_{\mathbf{R},i} \right) \otimes \left(\bigotimes_{j=1}^{b} \psi_{\mathbf{T},j} \right) \otimes \left(\bigotimes_{k=1}^{c} \psi_{\mathbf{Z},k} \right) \otimes \psi_F \right).$$

It is easy to see that the injection J is isometric. There exists a unique injection $\widehat{J} : L_2(\mathbf{R}, X) \to L_2(\widehat{H}, X)$, which is also an isometry, and for which the following diagram is commutative

$$\begin{array}{ccc} L_2(\mathbf{R}, X) & \xrightarrow{\mathcal{F}_{\mathbf{R}}} & L_2(\mathbf{R}, X) \\ J \downarrow & & \downarrow \widehat{J} \\ L_2(H, X) & \xrightarrow{\mathcal{F}_H} & L_2(\widehat{H}, X). \end{array}$$

Space $L_2(H, X)$ can be identified with a closed subspace of $L_2(G, X)$ consisting of functions, that are 0 almost everywhere outside H. Space $L_2(\widehat{H}, X)$ can be identified with a closed subspace of $L_2(\widehat{G}, X)$ consisting of functions, which are constant on cosets of $H^\perp$ (recall, that $\widehat{H} \cong \widehat{G}/H^\perp_G$).

Fourier transform $\mathcal{F}_H$ is the restriction of $\mathcal{F}_G$ and thus, bounded. Fourier transform $\mathcal{F}_{\mathbf{R}} = (\widehat{J})^{-1}\mathcal{F}_H J$ is continuous as a composition of continuous maps. By Theorem 1 statement 4) space X is isomorphic to a Hilbert one.

Cases when group G contains $\mathbf{T}$-component or $\mathbf{Z}$-component are considered similarly. ◀

Consider the case, when group G does not contain $\mathbf{R}$-, $\mathbf{Z}$- or $\mathbf{T}$-elements. Then all compactly generated subgroups $H \subset G$ are projective limits of finite subgroups with discrete topology, i.e. they are *profinite* groups.

Profinite groups are characterized by the following lemma [9].

Lemma 3 *Topological group H is a profinite one if and only if*

a) possesses Hausdorff's property;

b) is compact;

c) is totally disconnected, i.e. for any two points $x, y \in H$ there exists subset $U \subset H$ that is both open and closed, such that $x \in U$ and $y \notin U$.

Any profinite group H is either discrete (and finite by virtue of compactness) or non-discrete (and therefore infinite).

Consider non-discrete profinite group H. We normalize Haar measure on H with $\mu_H(H) = 1$.

Group H is a Lebesgue space, i.e. it is isomorphic as a space with measure to segment [0,1] with length [5, 8]. This fact can be proved selecting sequence of compact subgroups $K_n \subset H$ such that $K_1 \subset K_2 \subset ...$ and cardinality of quotient groups $M_n := |H/K_n|$ tends to $+\infty$. If one numbers cosets of K_n properly, he becomes able to identify them with the intervals of length $1/M_n$ in $[0, 1]$ up to a subsets of zero measure. By τ denote this isomorphism of spaces with measure.

A system of functions $(r_i)_{i=1,2,\dots}$, similar to the Rademacher's system on $[0,1]$ can be constructed on group H. This is an orthogonal system of functions taking values $\{+1,-1\}$ on subsets of measure $\frac{1}{2}$. Saying in probability-theoretical language functions r_i are realizations of independent random variables taking values $\{+1,-1\}$ with probability $\frac{1}{2}$. One can simply assume $r_i = r_i^\infty \circ \tau$, where $r_i^\infty(t) = \sin 2^i \pi t$ on $[0,1]$ is the usual Rademacher's functions.

We need a criterion, which is proved in [3].

Theorem 4 *The following statements are equivalent:*

1) Banach space X is isomorphic to a Hilbert one.

2) There exists constant $C>0$ such that for any finite collection of vectors $x_1, x_2, \dots, x_n \in X$ two-sided Khinchin's type inequality holds

$$C^{-1}\sum_{i=1}^{n}\|x_i\|^2 \le \mathrm{E}\Big\|\sum_{i=1}^{n} r_i x_i\Big\|^2 = \int_H \Big\|\sum_{i=1}^{n} r_i(t) x_i\Big\|^2 dt \le C\sum_{i=1}^{n}\|x_i\|^2,$$

where r_i are independent random variables taking values $\{+1,-1\}$ with probability $\frac{1}{2}$, and symbol E *denotes expectation.*

As in [3] we formulate a Lemma, which shows importance of the system (r_i) on H and allows us to consider arbitrary basis in $L_2(H)$ instead of (r_i). By dt denote Haar measure on H.

Lemma 4 *Let X be a Banach space, (f_i) be an orthonormal complete system in $L_2(H)$. Assume that for some $C>0$ and for any collection $x_1, x_2, \dots, x_n \in X$ there is inequality*

$$\int_H \Big\|\sum_{i=1}^{n} f_i(t) x_i\Big\|^2 dt \le C\sum_{i=1}^{n}\|x_i\|^2 \quad \left(\textit{resp., } C^{-1}\sum_{i=1}^{n}\|x_i\|^2 \le \int_H \Big\|\sum_{i=1}^{n} f_i(t) x_i\Big\|^2 dt\right).$$

Then for the same constant $C>0$ and for any collection $x_1, x_2, \dots, x_n \in X$ one also has

$$\int_H \Big\|\sum_{i=1}^{n} r_i(t) x_i\Big\|^2 dt \le C\sum_{i=1}^{n}\|x_i\|^2 \quad \left(\textit{resp., } C^{-1}\sum_{i=1}^{n}\|x_i\|^2 \le \int_H \Big\|\sum_{i=1}^{n} f_i(t) x_i\Big\|^2 dt\right).$$

Proof. As Rademacher's system (r_i) is orthonormal and (f_k) are complete, we can find for a given $\varepsilon>0$ an increasing sequences of indices (k_j), (m_j) and orthonormal sequence (h_j) such that

$$h_j = \sum_{k=k_j}^{k_{j+1}-1} (h_j, f_k)\cdot f_k, \quad \int_H |h_j(t) - r_{m_j}(t)|^2 dt < \frac{\varepsilon}{2^j}.$$

For a fixed n and fixed $x_1, x_2, \dots, x_n \in X$ we have

$$\int_H \Big\|\sum_{i=1}^{n} r_i(t)\cdot x_i\Big\|^2 dt = \int_H \Big\|\sum_{i=1}^{n} r_{m_j}(t)\cdot x_i\Big\|^2 dt.$$

By the triangle inequality

$$\Big(\int_H \Big\|\sum_{j=1}^{n} r_{m_j}(t)\cdot x_j\Big\|^2 dt\Big)^{1/2} \le \Big(\int_H \Big\|\sum_{j=1}^{n} (r_{m_j}(t) - h_j(t))\cdot x_j\Big\|^2 dt\Big)^{1/2} +$$

$$+\Big(\int_H \Big\|\sum_{j=1}^{n} h_j(t)\cdot x_j\Big\|^2 dt\Big)^{1/2} \le \sqrt{\varepsilon}\Big(\sum_{j=1}^{n}\|x_j\|^2\Big)^{1/2} + \Big(\int_H \Big\|\sum_{j=1}^{n} h_j(t)\cdot x_j\Big\|^2 dt\Big)^{1/2}.$$

As $1 = \|h_j\|^2 = \sum\limits_{k=k_j}^{k_{j+1}-1} |(h_j, f_k)|^2$, we get

$$\int_H \Big\| \sum_{j=1}^n h_j(t) \cdot x_j \Big\|^2 dt = \int_H \Big\| \sum_{j=1}^n \Big(\sum_{k=k_j}^{k_{j+1}-1} (h_j, f_k) \cdot f_k \Big) x_j \Big\|^2 dt \le$$

$$\le C \sum_{j=1}^n \sum_{k=k_j}^{k_{j+1}-1} |(h_j, f_k)|^2 \|x_j\|^2 = C \sum_{j=1}^n \|x_j\|^2.$$

Thus,

$$\int_H \Big\| \sum_{i=1}^n r_i(t) \cdot x_i \Big\|^2 dt \le (\sqrt{\varepsilon} + \sqrt{C})^2 \sum_{i=1}^n \|x_i\|^2.$$

As ε is arbitrary, we obtain the desired inequality. Proof in the case of reverse type inequality is analogous. ◄

Corollary 4.1 *Let X be a Banach space, and (f_i) be a complete orthonormal system in $L_2(H)$. Space X is linearly isomorphic to a Hilbert one if and only if there exists constant $C > 0$ such that for any set of vectors $x_1, x_2, \dots, x_n \in X$ one has*

$$C^{-1} \sum_{i=1}^n \|x_i\|^2 \le \int_H \Big\| \sum_{i=1}^n f_i(t) \cdot x_i \Big\|^2 dt \le C \sum_{i=1}^n \|x_i\|^2.$$

In Corollary 4.1 isomorphism of X to a Hilbert space follows from *two-sided* inequality. Knowledge of profinite groups' structure allows us to switch from lower estumate to upper one and vice versa as shown below.

First recall that Bruhat–Schwartz space on a profinite group H and on dual discrete $\widehat{H}$ consists of locally constant functions with compact support. Spaces $\mathcal{S}(H)$ and $\mathcal{S}(\widehat{H})$ are inductive limit of finite-dimensional spaces and carry the strongest locally convex topology [7].

Now we are going to study the case of vector-valued Fourier transform on a profinite non-discrete group H. As H is a compact infinite group, its dual $\widehat{H}$ is a discrete infinite group.

Lemma 5 *Let X be a Banach space, and H be a profinite non-discrete group. The following statements are equivalent:*

1) *X is linearly isomorphic to a Hilbert space.*

2) *There exists constant $C > 0$ such that for any set of vectors $x_1, ..., x_n \in X$ and characters $\xi_1, ..., \xi_n \in \widehat{H}$ one has*

$$\int_H \Big\| \sum_{k=1}^n \langle \xi_k, t \rangle x_k \Big\|^2 dt \le C \sum_{k=1}^n \|x_k\|^2.$$

2)′ *Fourier transform $\mathcal{F}_{\widehat{H}} : L_2(\widehat{H}, X) \to L_2(H, X)$ and inverse Fourier transform $F_H^{-1} = \mathcal{I}_H \mathcal{F}_{\widehat{H}}$ are bounded. Here $\mathcal{I}_H$ is an isometrical operator of changing variable $x(t) \mapsto x(-t)$.*

3) *There exists constat $C > 0$ such that for any set of vectors $x_1, ..., x_n \in X$ and characters $\xi_1, ..., \xi_n \in \widehat{H}$ one has*

$$C^{-1} \sum_{k=1}^n \|x_k\|^2 \le \int_H \Big\| \sum_{k=1}^n \langle \xi_k, t \rangle x_k \Big\|^2 dt.$$

3)$'$ *Inverse Fourier transform* $\mathcal{F}_{\widehat{H}}^{-1} : L_2(H,X) \to L_2(\widehat{H},X)$ *and Fourier transform* $\mathcal{F}_H = \mathcal{I}_H \mathcal{F}_{\widehat{H}}^{-1}$ *are bounded.*

Proof. Lemma 1 yields implications 1) $\Rightarrow$ 2)$'$, 1) $\Rightarrow$ 3)$'$.

To prove equivalences 2)$'$ $\Leftrightarrow$ 2), 3)$'$ $\Rightarrow$ 3) it's enough to consider

$$h = \sum_{k=1}^{n} I_{\{\xi_k\}} \cdot x_k \in \mathcal{S}(\widehat{H}) \otimes X,$$

where $\xi_k \in \widehat{H}, x_k \in X, n \in \mathbf{N}$. By definition of Fourier transform $\mathcal{F} : L_2(\widehat{H},X) \to L_2(H,X)$ and by chosen normalization of Haar measure on H we get

$$\|h\|^2_{L_2(\widehat{H},X)} = \sum_{k=1}^{n} \|x_k\|^2, \quad \|\mathcal{F}h\|^2_{L_2(H,X)} = \int_H \Big\| \sum_{k=1}^{n} \langle \xi_k, t\rangle x_k \Big\|^2 dt.$$

Equivalence follows from the density of $\mathcal{S}(\widehat{H}) \otimes X$ in $L_2(\widehat{H},X)$, density of $\mathcal{S}(H) \otimes X$ in $L_2(H,X)$, and bijectivity of $\mathcal{F}_{\widehat{H}}$, $\mathcal{F}_H$ in the corresponding spaces.

By Corollary 4.1 one also has 2) & 3) $\Rightarrow$ 1).

Now assume boundedness condition 2)$'$. Let's look at Fourier transform $\mathcal{F}_H$ on a subspace $\mathcal{S}(H) \otimes X$. For this consider arbitrary compact subgroup $K \subset H$, for which $|H/K| < +\infty$. We identify functions, which are constant on the cosets of K, with elements of $\mathcal{S}(H/K) \otimes X$.

By finiteness of H/K there exist an isomorphism $\alpha : \widehat{H/K} \to H/K$. Adjoint isomorphism $\alpha^* : \widehat{H/K} \to H/K$ is defined by

$$\langle \xi_1, \alpha(\xi_2)\rangle_H = \langle \alpha^*(\xi_1), \xi_2\rangle_{\widehat{H}}.$$

Consider operator

$$R_\alpha : \mathcal{S}(H/K) \otimes X \to \mathcal{S}(\widehat{H/K}) \otimes X : (R_\alpha \psi)(\xi') = \psi(\alpha(\xi')) \cdot |H/K|^{-\frac{1}{2}}.$$

It's an isometry, because

$$||R_\alpha \psi||^2 = \sum_{\xi' \in \widehat{H/K}} ||R_\alpha \psi(\xi)||^2 \mu_{\widehat{H/K}}(\xi) = \sum_{\xi' \in \widehat{H/K}} ||\psi(\alpha(\xi'))||^2 \cdot |H/K|^{-\frac{1}{2}\cdot 2} =$$
$$= [t := \alpha(\xi')] = \sum_{t \in H/K} ||\psi(t)||^2 \cdot |H/K|^{-1} = \sum_{t \in H/K} ||\psi(t)||^2 \mu_{H/K}(t) = ||\psi||^2.$$

Now one has

$$(\mathcal{F}_{\widehat{H}} R_\alpha \psi)(t') = \sum_{\xi' \in \widehat{H/K}} \langle t', \xi'\rangle_{\widehat{H/K}} (\psi(\alpha(\xi'))|H/K|^{-\frac{1}{2}}),$$

$$(R_{(\alpha^*)} \mathcal{F}_{\widehat{H}} R_\alpha \psi)(\xi) = \Big(\sum_{\xi' \in \widehat{H/K}} \langle \alpha^*(\xi), \xi'\rangle_{\widehat{H/K}} \psi(\alpha(\xi'))|H/K|^{-\frac{1}{2}} \Big) |H/K|^{-\frac{1}{2}} = [t := \alpha(\xi')] =$$
$$= \Big(\sum_{t \in H/K} \langle \alpha^*(\xi), \alpha^{-1}(t)\rangle_{\widehat{H/K}} \psi(t) \Big) |H/K|^{-1} = \sum_{t \in H/K} \langle \alpha^*(\xi), \alpha(\alpha^{-1}(t))\rangle_{H/K} \psi(t) |H/K|^{-1} =$$
$$= \sum_{t \in H/K} \langle \xi, t\rangle_{H/K} \psi(t) \mu_{H/K}(t) = (\mathcal{F}_H \psi)(\xi).$$

Thus, restriction of $\mathcal{F}_H$ onto $\mathcal{S}(H/K) \otimes X$ has the same norm as $\mathcal{F}_{\widehat{H}}$ does. As K is arbitrary, it implies continuity of $\mathcal{F}_H$ on $L_2(H, X)$, and implication 2) $\Rightarrow$ 3) is true. Implication 3) $\Rightarrow$ 2) can be proved in the same way.

Now one has enough implications to see the equivalence of all statements. ◀

Let's pass to the general case of Fourier transform on a locally compact Abelian group G. This is our main result.

Theorem 5 *Let X be a Banach space, let G be a locally compact Abelian group. Space X is linearly isomorphic to a Hilbert one if and only if Fourier transform*

$$\mathcal{F} : L_2(G, X) \to L_2(\widehat{G}, X)$$

is bounded.

Proof. Existence of isomorphism is a sufficient condition for boundedness of $\mathcal{F}$ as shown in Lemma 1.

Now assume that Fourier transform is bounded.

If G contains **R**-, **T**- or **Z**-component, then isomorphism of X to a Hilbert space follows from Lemma 2, and we are done.

Otherwise, group G does not contain **R**, **T** or **Z**-components. In this case all compactly generated open subgroups $H \subset G$ are profinite.

If some of these H is non-discrete, then we consider space $L_2(H, X)$ as a closed subspace in $L_2(G, X)$ (one can simply assume that function from $L_2(H, X)$ are zero outside H). We identify $L_2(\widehat{H}, X)$ with a subspace of $L_2(\widehat{G}, X)$ consisting of functions constant on cosets of annihilator $H_G^{\perp} \subset \widehat{G}$. Fourier transform on $L_2(H, X)$ is the restriction of Fourier transform from $L_2(G, X)$ and is bounded. Isomorphism of X to a Hilbert space follows from Lemma 5, statement 3)$'$. And we are done.

If all subgroups $H \subset G$ considered are discrete (their compactness implies finiteness), then by Theorem 2 group G is an *inductive* limit of discrete subgroups. By properties of Pontryagin duality (in the language of Cathegory theory one can say that passing to a dual group is an exact functor) dual group $\widehat{G}$ is a *projective* limit of $\widehat{H}$. Groups $\widehat{H}$ are dual to finite discrete H. Thus, $\widehat{H}$ are isomorphic H, and are finite discrete themselves. Group $\widehat{G}$ is profinite. As G is infinite, $\widehat{G}$ is non-discrete.

Boundedness of Fourier transform $\mathcal{F} : L_2(G, X) \to L_2(\widehat{G}, X)$ is equivalent to boundedness of inverse Fourier transform $\mathcal{F}_{\widehat{G}}^{-1}$ on profinite non-discrete $\widehat{G}$. Isomorphism of X to a Hilbert space follows from Lemma 5 statement 2)$'$. ◀

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